5.16

$u^{(i)} = h_i u^i$

~~$u^{(r)} = u^r$~~

~~$u^{(\phi)} = r u^{\phi}$~~

~~$u^{(z)} = u^z$~~

. . . .

written correctly, the physical components are

$$\left(dr/dt,\ r\,d\phi/dt,\ dz/dt \right)$$

$$\sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\phi}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \text{speed}$$

5.13

$$u^i a_i \cdot a_j = u_i a^i \cdot a_j$$

$$u^i a_i \cdot a_j = u_i \delta^i_j$$

$$u^i g_{ij} = u_i \delta^i_j \longleftarrow (i=j, \text{ else } 0)$$

$$u^i g_{ij} = u_j$$

5.14

$$u = u_j a^j = g_{ij} u^i a^j$$

$$u = u^i a_i$$

$$g_{ij} u^i a^j = u^i a_i$$

$$g_{ij} a^j = a_i$$

5.15

$$[g^{ij}] = [g_{ij}]^{-1}$$

For an orthgonal coordinate system, $[g_{ij}] = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$, which is a diagonal matrise. The inverse of a diagonal matrise is the matrise where every element of the diagonal matrise is raised to the power -1. (~~also~~ 1 divided by)

Hence $[g^{ij}] = \begin{pmatrix} \frac{1}{h_1^2} & 0 & 0 \\ 0 & \frac{1}{h_2^2} & 0 \\ 0 & 0 & \frac{1}{h_3^2} \end{pmatrix} = \begin{pmatrix} h_1^{-2} & 0 & 0 \\ 0 & h_2^{-2} & 0 \\ 0 & 0 & h_3^{-2} \end{pmatrix}$

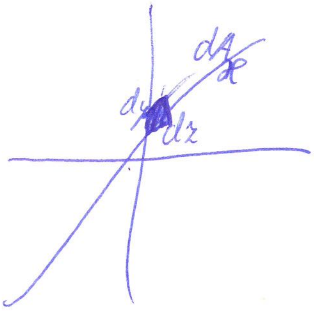
5.12

$$dA_1 = \|dA_1\| = \|a_2 \times a_3\| \, dx^2 dx^3$$

$$= \|a_2\| \; \|a_3\| \, dx^2 dx^3$$

$$= h_2 \; h_3 \, dx^2 dx^3$$

$$= (h_2 dx^2)(h_3 dx^3)$$

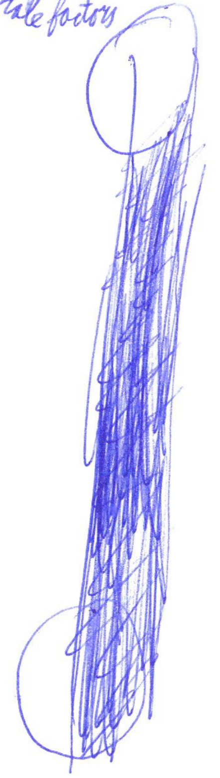


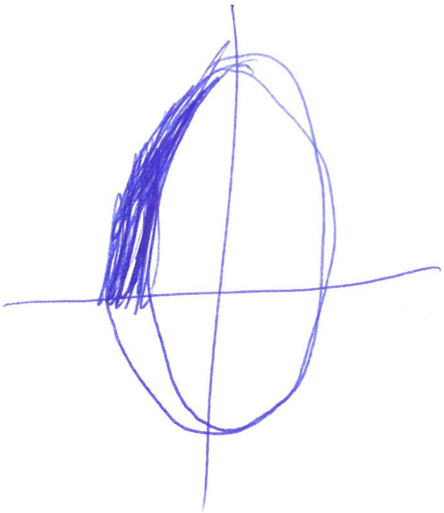
product of physical displacements → obtainable by scaling coordinate displacements with scale factors

Cartesian $dA_x =$ [illegible] $dy \, dz$

Cylindrical $dA_r = r \, d\phi \, dz$

Spherical $dA_r = r^2 \sin\theta$ [illegible] $d\theta \, d\phi$





5.10

Cartesian $dV =$ [redacted] $dx dy dz$

cylindrical $dV =$ [redacted] $dr \cdot r d\phi \cdot dz = r dr d\phi dz$

spherical $dV = dr \cdot r d\theta \cdot r \sin\theta d\phi = r^2 \sin\theta dr d\theta d\phi$

5.11

[redacted]

[redacted]

5.9

$$h_r = \sqrt{a_r \cdot a_r} = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta} = \sqrt{\sin^2\theta+\cos^2\theta} = 1$$

$$h_\theta = \sqrt{a_\theta \cdot a_\theta} = r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta} = r$$

$$h_\phi = \sqrt{a_\phi \cdot a_\phi} = r\sin\theta\sqrt{\sin^2\phi+\cos^2\phi} = r\sin\theta$$

5.10

$$[g_{ij}] = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$

$$\det(g_{ij}) = h_1^2 h_2^2 h_3^2$$

$$g = h_1^2 h_2^2 h_3^2$$

$$\sqrt{g} = h_1 h_2 h_3$$

$$dV = \sqrt{g}\, dx^1 dx^2 dx^3 = h_1 h_2 h_3\, dx^1 dx^2 dx^3$$

$$= (h_1 dx^1)(h_2 dx^2)(h_3 dx^3)$$

5.8

$$\left\|\frac{dx}{dt}\right\| = \sqrt{0^2 + (R\omega)^2 + \left(\frac{H}{2T}\right)^2}$$

$$= \sqrt{R^2\omega^2 + \frac{H^2}{4T^2}}$$

$$s(t) = \int_{t_0}^{t} \sqrt{R^2\omega^2 + \frac{H^2}{4T^2}}\, dt = \int_0^t \sqrt{R^2\omega^2 + \left(\frac{H}{2T}\right)^2}$$, which is the same result.

5.9

$$a_i = \frac{\partial}{\partial x^i}\underline{x}$$

$$a_r = \frac{\partial}{\partial r}\underline{x} = \sin\theta\cos\phi\,\underline{e_x} + \sin\theta\sin\phi\,\underline{e_y} + \cos\theta\,\underline{e_z}$$

$$a_\theta = \frac{\partial}{\partial \theta}\underline{x} =$$

$$r\cos\phi\cos\theta\,\underline{e_x} + r\sin\phi\cos\theta\,\underline{e_y} - r\sin\theta\,\underline{e_z} =$$

$$r(\cos\theta\cos\phi\,\underline{e_x} + \cos\theta\sin\phi\,\underline{e_y} - \sin\theta\,\underline{e_z})$$

$$a_\phi = \frac{\partial}{\partial \phi}\underline{x} = -r\sin\theta\sin\phi\,\underline{e_x} + r\sin\theta\cos\phi\,\underline{e_y}$$

$$= r\sin\theta\left(-\sin\phi\,\underline{e_x} + \cos\phi\,\underline{e_y}\right)$$

$$a_r \cdot a_\theta = \sin\theta\cos\phi \cdot r\cos\phi\cos\theta + \sin\theta\sin\phi \cdot r\sin\phi\cos\theta - r\sin\theta\cos\theta$$

$$= r\left(\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi\right) - r\sin\theta\cos\theta$$

$$= 0$$

$$a_r \cdot a_\phi = -r\sin\theta\cos\phi\sin\theta\sin\phi + r\sin\theta\cos\phi\sin\theta\sin\phi = 0$$

$$a_\theta \cdot a_\phi = -r^2\sin\theta\sin\phi\cos\theta\cos\phi + r^2\sin\theta\cos\phi\cos\theta\sin\phi = 0$$

Hence, it is orthogonal

5.6

$$\frac{dr}{dt} = \frac{\partial r}{\partial x^i} \frac{dx^i}{dt} = a_i \frac{dx^i}{dt}$$

why straight d's (not ∂'s)?

5.7

$$\left\| \frac{dx}{dt} \right\| = \sqrt{\frac{dx}{dt} \cdot \frac{dx}{dt}} = \sqrt{\left(a_i \frac{dx^i}{dt}\right) \cdot \left(a_i \frac{dx^i}{dt}\right)}$$

$$= \sqrt{\frac{dx^i}{dt} \frac{dx^i}{dt} g_{ii}}$$

$$= \sqrt{\sum_i \left(\frac{dx^i}{dt} \frac{dx^i}{dt} h_i^2\right)}$$

$$= \sqrt{\sum_i \left(h_i \frac{dx^i}{dt}\right)^2}$$

Copied from answers:

The coefficients h determine ('scale') the coordinate intervals dx^i to obtain 'physical' displacements/sizes $h\,dx^i$

5.4

z-coordinate line : a line parallel to the z- axis (of the cylinder, (e.g. a line over the surface of the cylinder))

z-coordinate surface : a surface/plane perpendicular to the z-axis

r-coordinate line : a straight line going away from the z-axis

! ! r-coordinate surface : cylinders with radius r

ϕ-coordinate line : a circle around the z-axis

ϕ-coordinate surface : (half-) planes ~~with radius~~ bounded by the z-axis

?

5.5

$$\underline{a}_r = \frac{\partial}{\partial r}\underline{x} = \cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y$$

$$\underline{a}_\phi = \frac{\partial}{\partial\phi}\underline{x} = -r\sin\phi\,\underline{e}_x + r\cos\phi\,\underline{e}_y$$

$$\underline{a}_z = \underline{e}_z$$

$$\underline{a}_r\cdot\underline{a}_\phi = \cos\phi\cdot -r\sin\phi + \sin\phi\cdot r\cos\phi = 0$$

$$\underline{a}_r\cdot\underline{a}_z = 0$$

$$\underline{a}_\phi\cdot\underline{a}_z = 0$$

Hence, this covariant basis is orthogonal

$$||\underline{a}_r|| = \cos^2\phi + \sin^2\phi = 1$$

$$||\underline{a}_\phi|| = r^2(-\sin^2\phi) + r^2\cos^2\phi = r^2 \ \#$$

$$||\underline{a}_z|| = 1$$

Hence, this covariant basis is orthonormal iff $r^2 = 1$

5.1

$$\underline{e}_r = C_r \frac{\partial \mathbf{r}}{\partial r}\bigg|_\phi = C_r\left(\cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y\right)$$

$$= +\cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y$$

$$= +\underline{e}_x\cos\phi + \underline{e}_y\sin\phi$$

$$\underline{e}_\phi = C_\phi \frac{\partial \mathbf{r}}{\partial \phi}\bigg|_r = C_\phi\left(-\underline{e}_x\, r\sin\phi + \underline{e}_y\, r\cos\phi\right)$$

$$C_\phi = \frac{1}{\mathbf{r}}$$

sqrt ([illegible]) = r
dot prod
with self

$$= -\underline{e}_x\sin\phi + \underline{e}_y\cos\phi$$

5.2

$$x(r,\phi) = e_x\, x(r,\phi) + e_y\, y(r,\phi)$$

$$= \underline{e}_x\, r\cos\phi + \underline{e}_y\, r\sin\phi$$

$$= r\left(\cos\phi\,\underline{e}_x + \sin\phi\,\underline{e}_y\right)$$

$$= r\,e_r(\phi)$$

5.3/1

$$\frac{d\underline{e}_r(\phi)}{d\phi} = -\underline{e}_x\sin\phi + e_y\cos\phi = \underline{e}_\phi(\phi)$$

5.3/2

$$\frac{d\underline{r}}{dt} = \frac{d(\underline{r}\,\underline{e}_r(\phi))}{dt} = \frac{dr}{dt}\underline{e}_r(\phi) + r\cdot\frac{d\underline{e}_r(\phi)}{dt} =$$

$$\frac{dr}{dt}e_r(\phi) + r\cdot\frac{de_r(\phi)}{d\phi}\frac{d\phi}{dt} =$$

$$\frac{dr}{dt}e_r + \left(r\frac{d\phi}{dt}\right)e_\phi$$

4.9

$$\underline{x}'(t) = -R\omega \sin(\omega t)\,\underline{e}_x + R\omega\cos(\omega t)\,\underline{e}_y$$

$$\underline{x}'(t(u)) = 2uTR\omega\left(-\sin(\omega t)\,\underline{e}_x + \cos(\omega t)\,\underline{e}_y\right)$$

$$u = \sqrt{\frac{t}{T}}$$

$$u^2 = \frac{t}{T}$$

$$t = u^2 T$$

$$\frac{dt}{du} = 2uT$$

$$\|\underline{x}'(t)\| = R\omega$$

$$\|x'(t(u))\| = 2uTR\omega = 2uTR\frac{2\pi}{T} = 4\pi u R$$

4.10

1

speed being unity implies

$$\left\|\frac{d\underline{x}(t(u))}{du}\right\| = 1$$

$$\left\|\frac{dt}{du}\frac{d\underline{x}(t)}{dt}\right\| = 1$$

$$\sqrt{\left(\frac{dt}{du}\frac{d\underline{x}(t)}{dt}\right)^2} = 1$$

$$\frac{dt}{du}\frac{d\underline{x}t}{dt} = 1$$

$$\frac{d\underline{x}(t)}{dt} = \left(\frac{dt}{du}\right)^{-1}$$

which implies $\left\|\frac{d\underline{x}(t)}{dt}\right\| = \left\|\frac{dt}{du}\right\|^{-1}$

$$1 = \frac{d\underline{x}(t(u))}{du} = \left\|\frac{dt}{du}\right\|\,\left\|\frac{d\underline{x}(t)}{dt}\right\|$$

$$\left\|\frac{dt}{du}\right\|^{-1} = \left\|\frac{d\underline{x}(t)}{dt}\right\|$$

?

$$\left|\frac{dt}{du}\right|^{-1}$$

why modulus?!

?!

$\boxed{\frac{4.10}{2}}$

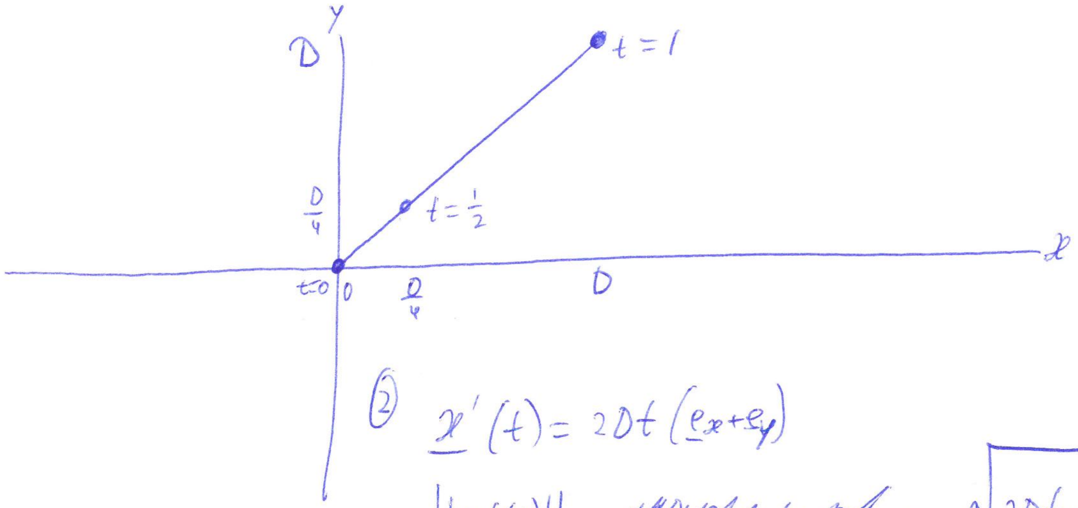
$$\left(\frac{dt}{du}\right)^{-1} = \frac{du}{dt} = \|x'(t)\|$$

$$s(t) = \int_{t_a}^{t} \|x'(t)\|\,dt = \int_{t_a}^{t} \frac{du}{dt}\,dt = \int_{t_a}^{t} du = u(t)$$

(as u ranges from ta to tb, for instance)

$\boxed{4.11}$

①



②

$$x'(t) = 2Dt\,(e_x + e_y)$$

$$\|x'(t)\| = \sqrt{2Dt\cdot 2Dt + 2Dt\cdot 2Dt} = \sqrt{2}\cdot 2Dt$$

$$s(t) = \int_0^t 2\sqrt{2}Dt\,dt = \frac{2}{2}\sqrt{2}\,D\,t^2 = \sqrt{2}Dt^2$$

③

$$s_a = s(0) = 0$$

$s_b = s(1) = \sqrt{2}D$, which is the curve length (by Pythagoras)

④

$$s = \sqrt{2}\,Dt^2$$

$$\frac{s}{\sqrt{2}D} = t^2 \longrightarrow t = \sqrt{\frac{s}{\sqrt{2}D}}$$

$$x_s(s) = x(t(s)) = D\,\frac{s}{\sqrt{2}D}\,(e_x + e_y) = s\,\frac{e_x + e_y}{\sqrt{2}}$$

⑤

$s_a = s(0) = 0$, $x_s(s_a) = (0,0)$

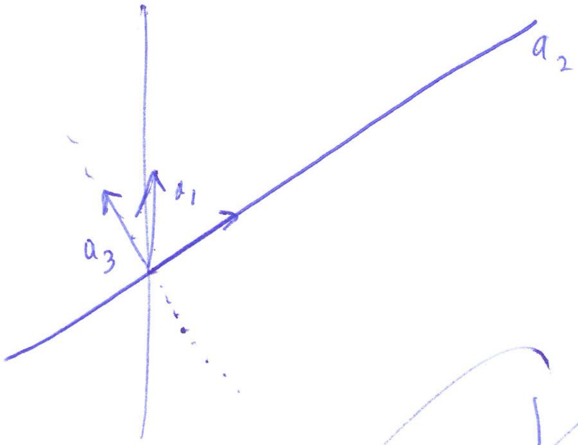
$s_b = s(1)$, $x_s(s_b) = \frac{e_x + e_y}{\sqrt{2}} \rightarrow D(1,1)$

$$x\left(\frac{s_a + s_b}{2}\right) = s\left(\frac{0+1}{2}\right) = s\left(\frac{1}{2}\right) = \frac{e_x + e_y}{2\sqrt{2}} = \frac{1}{2}\,\frac{e_x + e_y}{\sqrt{2}},$$

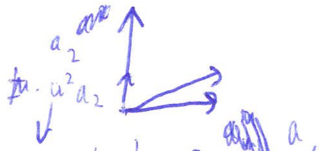
$\uparrow$ which is exactly halfway in between s_a and s_b

$D\left(\frac{1}{2}, \frac{1}{2}\right)$

2.17



2.15 ③



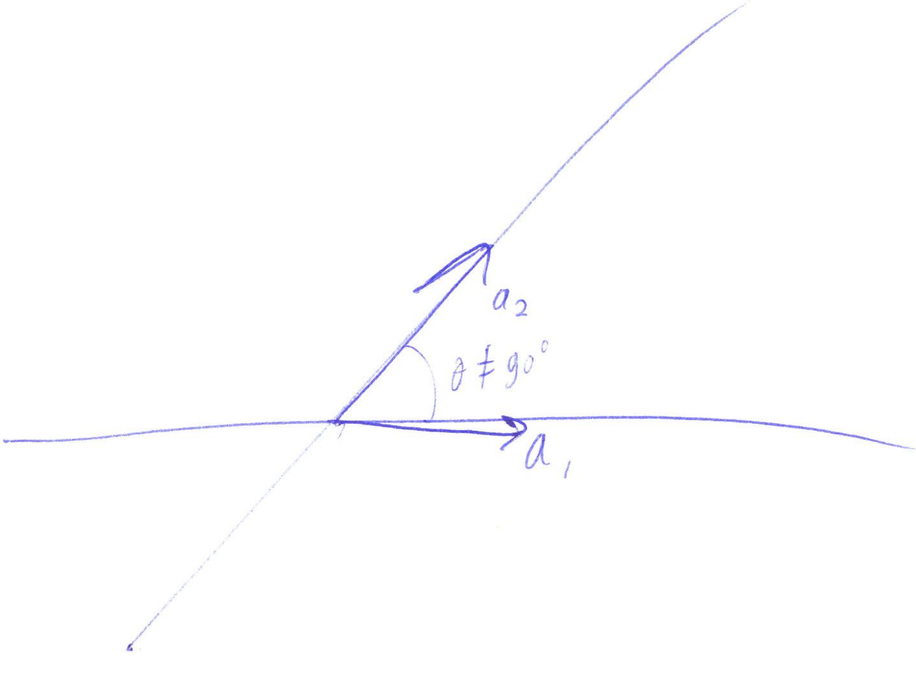
$||u^2 a_2|| = |u^2|$

$u^1 a_1$

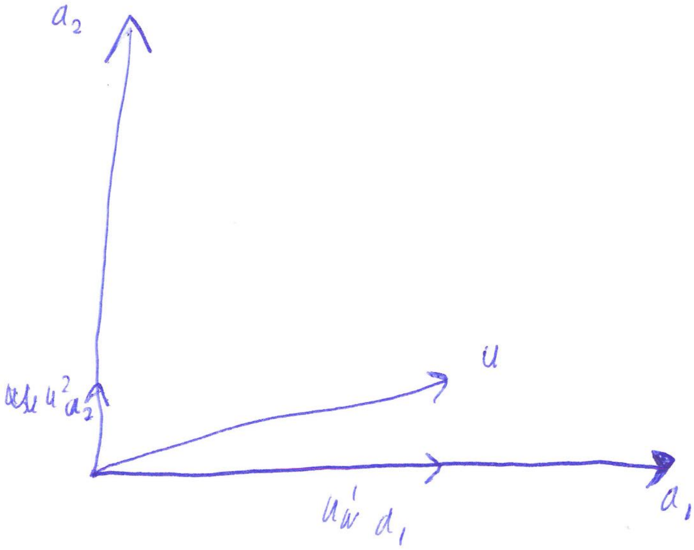
↳ $||u^1 a_1|| = |u^1|$

$$||u|| = \sqrt{(u^1)^2 + (u^2)^2}$$

2.16



2.14 ③



Remember:

projections are $u^1 a_1 \rightarrow |u^1 a_1| = \sqrt{(u^1 h_1)^2} = |u^1 h_1|$

and $u^2 a_2 \rightarrow |u^2 a_2| = \sqrt{(u^2 h_2)^2} = |u^2 h_2|$

$$\|u\| = \sqrt{(u^1 h_1)^2 + (u^2 h_2)^2}$$

(as projections are orthogonal)

2.15

①

[redacted]

$g_{ij} = 0$ if $i \neq j$ because of orthogonality

$g_{ij} = 1$ if $i = j$ because $g_{ii} = h_i^2 = \|a_i\| = 1^2 = 1$

Hence $g_{ij} = \delta_{ij}$

② $U \cdot V = u^i v^j \delta_{ij} = \sum_i u^i v^i$ (as $u^i v^j \delta_{ij} = 0$ if $i \neq j$)

$$\|u\| = \sqrt{u \cdot u} = \sqrt{\sum_i u^i u^i} = \sqrt{\sum_i (u^i)^2}$$ #

2. 13

[redacted]

① $g_{ij} = a_i \cdot a_j = a_j \cdot a_i = g_{ji}$ by commutatio of the dot product

② if $n=3$, g_{ij} has 6 independent elements (symmetry; only diagona and above it count

2. 14

①

$$g_{ij} = \begin{pmatrix} h_1^2 & 0 & 0 \\ 0 & h_2^2 & 0 \\ 0 & 0 & h_3^2 \end{pmatrix}$$, which is a diagonal matrix

② $u \cdot v = u^i v^j g_{ij}$ (as $g_{ij} = 0$ if $i \neq j$)

$= u^i v^i g_{ii}$ (as $g_{ii} = h_i^2$)

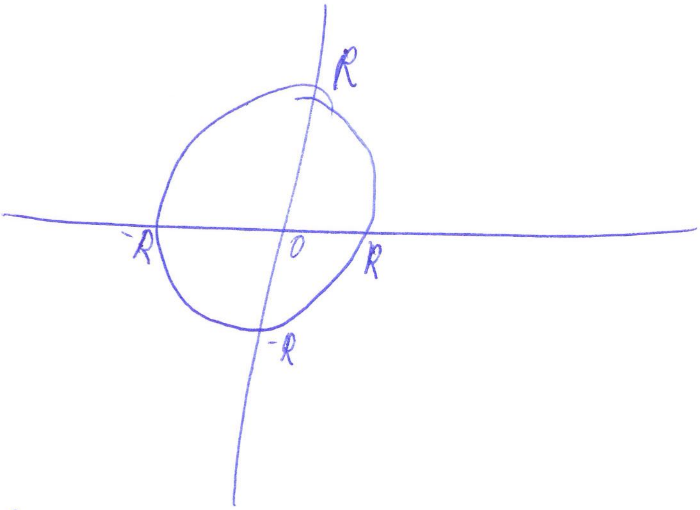
$= \sum_i u^i v^i (h_i)^2$

also, $\|u\| = \sqrt{u \cdot u} = \sqrt{u^i u^i g_{ii}}$

$= \sqrt{\sum_i (u^i)^2 h_i^2}$

$= \sqrt{\sum_i (u^i h_i)^2}$

4.1 ①

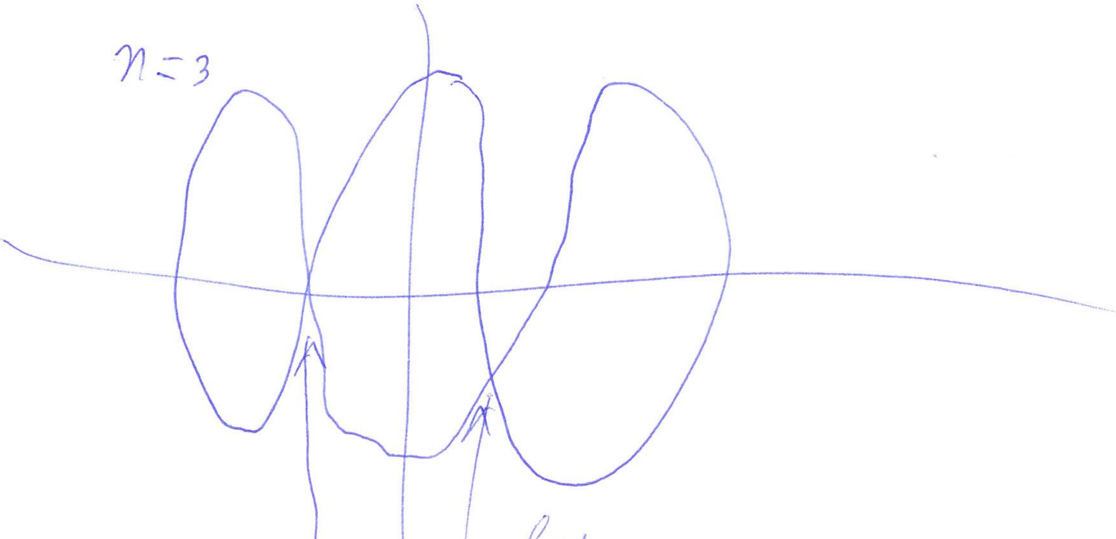


left- and right-closed, simple

②

~~left-~~

$n = 3$



left- and right-closed

non-simple (intersects itself)

3.1

$p = m x'(t)$

$L = x(t) \times m x'(t) = m x \times x' \quad (= x \times p)$

$F = m x''(t)$

$E = \frac{1}{2} m (x')^2$

$\mathcal{E}' = \left(\frac{1}{2} m (x')^2\right)' = \frac{1}{2} m \cdot 2x' \cdot x'' = m x' \cdot x'' = m x'' \cdot x' = x' \cdot F$

$q x' \times B$ is orthogonal to x', hence $\mathcal{E}' = x' \cdot F = x' \cdot (q x' \times B) = 0$, so the force does not affect the kinetic energy of the particle

$$L' = (m x \times x')' = m x \times x'' + \overbrace{m x' \times x'}^{=0} = x \times m x'' = r \times F$$

3.2

$$\frac{\partial x}{\partial x} = \underbrace{\frac{\partial x^i}{\partial x}}_{=0 \text{ if } x^i \neq x} \underline{e_i} + x^i \frac{\partial e_i}{\partial x}$$

In Cartesian system, e_i is constant, hence $= 0$

$$= \frac{\partial x}{\partial x} \underline{e_x} + 0$$

$$= \underline{e_x}$$

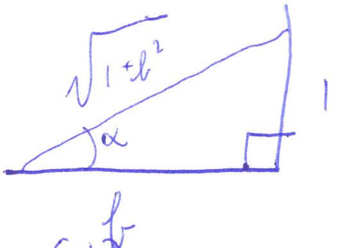
4.5

① $x \cdot x' = \|x\| \|x'\| \cos\alpha = b\|x\|^2$

$\|x'\| \cos\alpha = b\|x\|$

$\|x\|\sqrt{1+b^2} \cos\alpha = b\|x\|$

$$\cos\alpha = \frac{b}{\sqrt{1+b^2}}$$



$$\alpha = \arctan\left(\frac{1}{b}\right)$$

4.6

$s(t) = \int_0^{t} \|x'\| dt =$ ~~[illegible]~~ ~~$= \int_0^T \sqrt{R^2}\, dt$~~

~~$= \int_0^T R\,dt = RT = 2\pi R$~~

$$\int_0^{t} \sqrt{R^2 w^2 \sin^2(wt) + R^2 w^2 \cos^2 wt}\; dt$$

$$= \int_0^{t} Rw\,dt = Rw t = R\left(\frac{2\pi}{T}\right)t$$

$s(T) = 2\pi R$, which is expected for the circumference of a circle

4.7

$$s(t) = \int_0^t \|x'\| dt = \int_0^t \sqrt{R^2 w^2 \sin^2(wt) + R^2 w^2 \cos^2(wt) + \left(\frac{H}{2T}\right)^2} \; dt$$

$$= \int_0^t \sqrt{R^2 w^2 + \left(\frac{H}{2T}\right)^2} \; dt =$$

4.8

(a) $$s(\phi) = \int_0^\phi \|x'\| d\phi = \int_0^\phi \|x\| \sqrt{1+b^2} \, d\phi = \int_0^\phi a e^{b\phi} \sqrt{1+b^2} \, d\phi$$

$$= a\sqrt{1+b^2} \int_0^\phi e^{b\phi} d\phi$$

$$= \frac{a}{b} \sqrt{1+b^2} \left[e^{b\phi} \right]_0^\phi$$

$$= \frac{a}{b} \sqrt{1+b^2} \, e^{b\phi}$$

$$= \frac{r}{b} \sqrt{1+b^2} \; .$$

$$= \frac{r}{\cos(\alpha)}$$

4.2)

a curve of which the 'height' in the z-direction is increasing with a constant rate as the (x, y) - position moves on a circle with radius R.

Answer: helix with height H and radius R with 2 revolutions

4.3₁

$$\underline{V} = -R\omega \sin(\omega t)\underline{e_x} + R\omega \cos(\omega t)\underline{e_y}$$

$$||\underline{V}|| = \sqrt{(-R\omega \sin(\omega t))^2 + (R\omega \cos(\omega t))^2}$$

$$= \sqrt{R^2\omega^2} = R\omega$$

4.3₂

$$\underline{V} = -R\omega \sin(\omega t)\underline{e_x} + R\omega n \cos(n\omega t)\underline{e_y}$$

$$||\underline{V}|| = \sqrt{(-R\omega \sin(\omega t))^2 + (R\omega n \cos(n\omega t))^2}$$

$$= \omega R\sqrt{\sin^2(\omega t) + n^2\cos^2(n\omega t)}$$

4.4)

$$\underline{V} = \omega R \cos(\omega t)\vec{e_x} + \omega R \sin(\omega t)\vec{e_y} + H/(2T)\vec{e_z}$$

$$||\underline{V}|| = \sqrt{(\omega R\cos(\omega t))^2 + (\omega R \sin(\omega t))^2 + \left(\frac{H}{2T}\right)^2}$$

$$= \sqrt{\omega^2 R^2 + \frac{H^2}{4T^2}}$$

4.5

①

$$\|\underline{x}\| = \sqrt{\underline{x} \cdot \underline{x}}$$

$$= \sqrt{\left(ae^{b\phi} \cos\phi\right)^2 + \left(ae^{b\phi} \sin\phi\right)^2}$$

$$= \sqrt{\left(ae^{b\phi}\right)^2}$$

$$= ae^{b\phi}$$

②

$$x \cdot x' = x \cdot bx$$

$$= b\|x\|^2$$

(because second term is orthogonal to x; $\cos \cdot -\sin + \cos \cdot \sin$)

②

$$x' = b \cdot ae^{b\phi}\left(\cos(\phi)e_x + \sin(\phi)e_y\right) + ae^{b\phi}\left(-\sin\phi\, e_x + \cos\phi\, e_y\right)$$

$$= b\underline{x} + ae^{b\phi}\left(-e_x \sin\phi + e_y \cos\phi\right)$$

③

~~$\|x'\| = \sqrt{x' \cdot x'}$ $= \sqrt{\left(bae^{b\phi}\cos\phi - ae^{b\phi}\sin\phi\right)^2 + \left(bae^{b\phi}\sin\phi + ae^{b\phi}\cos\phi\right)}$~~

~~$= \sqrt{bae^{b\phi}}$~~

$$\|x'\|^2 = \underline{x'} \cdot \underline{x'} = \left(b\underline{x} + ae^{b\phi}\left(-\underline{e_x}\sin\phi + e_y \cos\phi\right)\right) \cdot \underline{x'}$$

$$= b\underline{x} \cdot \underline{x'} + ae^{b\phi}\left(-\underline{e_x}\sin\phi + \underline{e_y}\cos\phi\right) \cdot x'$$

$$= b^2\|\underline{x}\|^2 + ae^{b\phi}\left(-\underline{e_x}\sin\phi + \underline{e_y}\cos\phi\right) \cdot \left(\overset{=0}{\cancel{b\underline{x}}} + ae^{b\phi}\left(-e_x\sin\phi + e_y\cos\phi\right)\right)$$

$$= b^2\|x\|^2 + \left(ae^{b\phi}\right)^2 = b^2\|\underline{x}\|^2 + \|x\|^2 = \|x\|^2(1+b^2) \to \text{sqrt gives result}$$

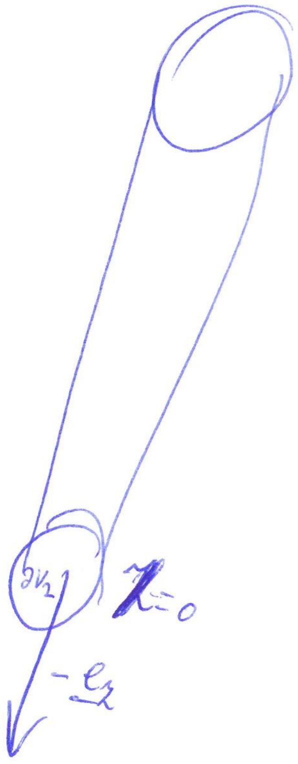
$$\ldots^2 = 1$$

$\boxed{6.15}$ $\operatorname{div} v = \frac{1}{r} \frac{\partial rz}{\partial z} = \frac{\partial z}{\partial z} = 1$

$$\int_{\partial V_1} (v \cdot n)\, dA = \int_V \nabla \cdot v \, dV - \int_{\partial V_2} (v \cdot n)\, dA$$

$$= \int_V dV - \int_{\partial V_2} (v \cdot n)\, dA$$

$$= \frac{1}{3}\pi R^2 H - \int_{\partial V_2} (v \cdot n)\, dA$$



in plane at $z=0$, $n = -e_z$, hence

$$v \cdot n = z e_z \cdot - e_z$$
$$= -z$$

$$\int_{\partial V_1} (v \cdot n)\, dA = \frac{1}{3}\pi R^2 H - \int_{\partial V_2} -z\, dA$$

$z=0$

$$= \frac{1}{3}\pi R^2 H - \cancel{(-z \cdot \pi R^2)}$$
$$= \frac{1}{3}\pi R^2 H \cancel{+ z \pi R^2}$$

$$= \pi R^2 \left(\frac{1}{3} H \cancel{+ z}\right)$$

∂V_2 has surface area $\pi \cdot R^2$

Then, $z \in [0, R]$ gives $H = R$, so the flux is equal to $\frac{1}{3}\pi R^3$

6.14

$$\frac{d}{dt}\int_V n_s(\underline{x})\,dV + \int_{\partial V}(\Gamma_s \cdot \underline{n})\,dA = \int_V S_s(\underline{x})\,dV$$

volume is fixed ⇒ ⇒ time-derivative can be taken inside integral; partial derivative

$$\int_V \frac{\partial}{\partial t} n_s(\underline{x})\,dV + \int_V \nabla \cdot \Gamma\,dV - \int_V S_s(\underline{x})\,dV = 0$$

$$\int_V \left(\frac{\partial}{\partial t} n_s(\underline{x}) + \nabla \cdot \Gamma - S_s(\underline{x})\right) dV = 0$$

6.12

What is 'the' position vector $\underline{x}$?

Cartesian: $\underline{x} = x\underline{e}_x + y\underline{e}_y + z\underline{e}_z$

$$\nabla \cdot \underline{x} = 1+1+1 = 3$$

Cylindrical: $\underline{x} = r\underline{e}_r^{\,(r,\phi)} + z\underline{e}_z$

$$\nabla \cdot \underline{x} = \frac{1}{r}\left(\frac{\partial r \cdot r}{\partial r} + \frac{\partial r z}{\partial z}\right)$$

$$= \frac{1}{r}\frac{\partial r^2}{\partial r} + \frac{\partial z}{\partial z}$$

$$= \frac{2r}{r} + 1$$

$$= 2 + 1 = 3$$

Spherical: $\underline{x} = r\underline{e}_r\,(r,\theta,\phi)$

$$\nabla \cdot \underline{x} = \frac{1}{r^2 \sin\theta}\left(\frac{\partial r^2 \sin\theta\, r}{\partial r}\right)$$

$$= \frac{1}{r^2}\frac{\partial r^3}{\partial r}$$

$$= \frac{1}{r^2} \cdot 3r^2$$

$$= 3$$

↳ all results are equal

6.13

This is the electric field of a point charge in a point at distance r from it. The source is a point charge at $r = 0$

(Divergence = 0 at $r > 0$, as there is no source of charge there)

6.11 ①

$$\operatorname{div} v = \frac{1}{1\cdot 1\cdot 1}\left(\frac{\partial 1\cdot 1\cdot x^3}{\partial x} + 3y^2 + 3z^2\right)$$

$$= 3\left(x^2+y^2+z^2\right)$$

②

$$\operatorname{div} \mathbf{v} = \frac{1}{r}\left(\frac{\partial r^2\cos^2\phi}{\partial r} + \frac{\partial r\, 2z\sin^2\phi}{\partial z}\right)$$

$$= \frac{1}{r}\left(2r\cos^2\phi + 2r\sin^2\phi\right)$$

$$= 2$$

③

$$\operatorname{div} v = \frac{1}{r^2\sin\theta}\left(\frac{\partial r^2\sin\theta\, r^{-n}}{\partial r}\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial r^{2-n}\sin\theta}{\partial r}\right)$$

$$= \frac{1}{r^2}\frac{\partial r^{2-n}}{\partial r}$$

$$= \frac{1}{r^2}(2-n)\, r^{1-n}$$

$$= r^{-n-1}\cdot(2-n)$$

$$= \frac{2-n}{r^{n+1}}$$

$$= \frac{1}{r^2}\frac{\partial r^2\cdot r^{-n}}{\partial r}$$

$$= \frac{1}{r^2}\left(r^{-n}\cdot 2r + (-n)\cdot r^2\cdot r^{(-n-1)}\right)$$

$$= \left(2r^{-n-1} - (n)\, r^{-n-1}\right)$$

$$= (2-n)\, r^{-n-1}$$

$$= \frac{2-n}{r^{n+1}}$$

6.10

Cartesian: $h_x = h_y = h_z = 1$

$$\nabla \cdot \Gamma = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial(h_2 h_3 \Gamma^{(1)})}{\partial x^1} + \frac{\partial(h_1 h_3 \Gamma^{(2)})}{\partial x^2} + \frac{\partial(h_2 h_3 \Gamma^{(3)})}{\partial x^3}\right)$$

$$\nabla \cdot \Gamma = \left(\frac{\partial \Gamma^{(x)}}{\partial x} + \frac{\partial \Gamma^{(y)}}{\partial y} + \frac{\partial \Gamma^{(z)}}{\partial z}\right)$$

$h_r = 1,\ h_\phi = r,\ h_z = 1$

Cylindrical
$$\nabla \cdot \Gamma = \frac{1}{r}\left(\frac{\partial r \cdot 1 \cdot \Gamma^{(r)}}{\partial r} + \frac{\partial 1 \cdot 1 \cdot \Gamma^{(\phi)}}{\partial \phi} + \frac{\partial r \cdot 1 \cdot \Gamma^{(z)}}{\partial z}\right)$$

$$= \frac{1}{r}\frac{\partial r \Gamma^{(r)}}{\partial r} + \frac{\partial \Gamma^{(\phi)}}{\partial \phi} + \frac{\partial \Gamma^{(z)}}{\partial z}$$

Cannot be taken outside partial derivative; is not constant

Spherical

$h_r = 1$
$h_\phi = r \sin\theta$
$h_\theta = r$

$$\nabla \cdot \Gamma = \frac{1}{r^2 \sin\theta}\left(\frac{\partial r \cdot r \cdot \sin\theta \cdot \Gamma^{(r)}}{\partial r} + \frac{\partial r \Gamma^{(\phi)}}{\partial \phi} + \frac{\partial r \sin\theta \Gamma^{(\theta)}}{\partial \theta}\right)$$

$$= \frac{1}{r^2}\frac{\partial r^2 \Gamma^{(r)}}{\partial r} + \frac{1}{r \sin\theta}\frac{\partial \Gamma^{(\phi)}}{\partial \phi} + \frac{1}{r \sin\theta}\frac{\partial \sin\theta \Gamma^{(\theta)}}{\partial \theta}$$

6.8 ③ The net flux through the x^2 and x^3 coordinate surfaces is obtained in an identical way. Hence, the total surface flux integral is:

$$\int_{\partial V} \left(\Gamma \cdot n\right) dA \approx \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\,\Gamma^i}{\partial x^i} \partial V$$

as $\partial x, \partial y, \partial z$ tend to zero, we get

$$\nabla \cdot \Gamma = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} \left(\Gamma \cdot n\right) dA = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\,\Gamma^i}{\partial x^i}$$

.9

$$\nabla \cdot \Gamma = \frac{1}{\sqrt{g}} \frac{\partial\left(\sqrt{g}\,\Gamma^i\right)}{\partial x^i} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial\left(h_1 h_2 h_3 \Gamma^1\right)}{\partial x^1} + \frac{\partial\left(h_1 h_2 h_3 \Gamma^2\right)}{\partial x^2} + \frac{\partial\left(h_1 h_2 h_3 \Gamma^3\right)}{\partial x^3} \right)$$

$$= \frac{1}{h_1 h_2 h_3} \left(\frac{\partial\left(h_2 h_3 \Gamma^{(1)}\right)}{\partial x^1} + \frac{\partial\left(h_1 h_3 \Gamma^{(2)}\right)}{\partial x^2} + \frac{\partial\left(h_1 h_2 \Gamma^{(3)}\right)}{\partial x^3} \right)$$

as $\Gamma^{(i)} = h_i \Gamma^i$

6.6

$$S_s(x) = \nabla \cdot \underline{\Gamma}$$

6.7

$$\int_{\partial V} (\underline{\Gamma}_s \cdot n)\, dA = \int_V (\nabla \cdot \underline{\Gamma})\, dV$$

6.8

①

$$\Gamma \cdot \delta A_1 = \Gamma \cdot \sqrt{g}\, a^1 \delta x^2 \delta x^3$$

$$= \Gamma^i a_i \cdot \sqrt{g}\, a^1 \delta x^2 \delta x^3$$

$$= \Gamma^i \delta_i^1 \sqrt{g}\, \delta x^2 \delta x^3$$

$$= \sqrt{g}\, \Gamma^1 \delta x^2 \delta x^3$$, which holds for both points

②

$$\sqrt{g}\,\Gamma x^1(x^1+\delta x^1, x^2, x^3) \overset{\delta x^2 \delta x^3}{} - \sqrt{g}\,\Gamma x^1(x^1, x^2, x^3)\,\delta x^2 \delta x^3 =$$

$$\left(\frac{\sqrt{g}\,\Gamma x^1(x^1+\delta x^1, x^2, x^3) - \sqrt{g}\,\Gamma x^1(x^1, x^2, x^3)}{\delta x^1}\right) \delta x^1 \delta x^2 \delta x^3 =$$

$$\frac{\partial \sqrt{g}\,\Gamma x^1}{\partial x^1} \qquad \frac{\delta V}{\sqrt{g}} =$$

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}\,\Gamma^1}{\partial x^1} \delta V$$

6.5

$$\Gamma_s = n_s V_s$$

The change in the number of particles in a given volume is the sum of the particles entering the volume, the particles produced inside the ~~sa~~ volume

Hence

$$\frac{d}{dt}\int_V n_s(x)\,dV = -\int_{\partial V}(\Gamma_s \cdot n)\,dA + \int_V S_s(x)\,dV$$

$$\frac{d}{dt}\underbrace{\int_V n_s(x)\,dV}_{\text{change (per unit of time) of num of particles}} + \underbrace{\int_{\partial V}(\Gamma_s \cdot n)\,dA}_{\text{particles entering volume (through area)}} = \underbrace{\int_V s_s(x)\,dV}_{\text{particles being formed in the volume}}$$

What should be 'derived'?

$\boxed{6.1}$

$$dV = dx \cdot dA = v(x)dt \cdot dA = v(x) \cdot dA\,dt$$

$$dm = \rho\,dV = \rho(v \cdot dA)\,dt$$

$$d\phi = \frac{dm}{dt} = \rho v \cdot dA$$

$$\frac{\delta m}{\delta t}$$

$\boxed{6.4}$

$$\frac{d}{dt}\int_V \rho(\underline{x})\,dV = -\int_{\partial V} (\rho\,\underline{v} \cdot \underline{n})\,dA$$

$$\frac{d}{dt}\rho V = -\int_{\partial V} (\rho\,\underline{v} \cdot \underline{n})\,dA$$

$$\cancel{\frac{dm}{dt} = \int_{\partial V} (\rho\,\underline{v} \cdot \underline{n})\,dA}$$

$$\frac{dV}{dt} = -\int_{\partial V} (\underline{v} \cdot \underline{n})\,dA$$

$$0 = -\int_{\partial V} (\underline{v} \cdot \underline{n})\,dA$$

$$\int_{\partial V} (\underline{v} \cdot \underline{n})\,dA = 0$$

7.13

to prove: $\frac{d}{dt}\left(\varepsilon\left(x'(t)\right) + U(x(t))\right) = 0 \left(\overset{\text{is the same as}}{\underset{\text{not bi-implication}}{\not\Longleftrightarrow}} \varepsilon\left(x'(t)\right) + U(x(t)) = \text{constant} \right)$; e.g. not to prove

$$\frac{d}{dt}\left(\varepsilon + U\right) = \frac{d\varepsilon}{dt} + \frac{dU}{dt} = P - P = 0$$, for right side, integrate

$\boxed{7.11}$

$$\frac{d\mathcal{E}}{dt} = \frac{1}{2} m \left(x' \cdot x'' + x'' \cdot x' \right)$$

$$= m \left(x' \cdot x'' \right)$$

$$= m x'' \cdot x'$$

$$= F \cdot x'$$

$$= P$$

$$x' \cdot x'$$

$$x' \cdot (x')' + x' \cdot (x')'$$

$$\nabla f \cdot x'(t) = \frac{df(x(t))}{dt}$$

$\boxed{7.12}$

to prove: $P(t) = -\frac{dU(x(t))}{dt}$

we have: $F = -\nabla U$

$$P(t) = \frac{dW}{dt} = F(x) \cdot x'(t) \, dt$$

$$= -\nabla U \cdot x'(t) \, dt$$

$$= -\frac{dU(x(t))}{dt}$$

$$\frac{dU}{dt} = \frac{\partial U}{\partial x^i} a^i \cdot \frac{dx}{dt} = \frac{\partial U}{\partial x^i} a^i \cdot \frac{dx^j}{dt} a_j =$$

$$\frac{\partial U}{\partial x^i} a^i \cdot x'(t) =$$

$$\nabla U \cdot x'(t)$$

7.10 $u(x) = \nabla f(x) = df$

dividing 7.3 by dt:

$$\frac{df}{dt} = \frac{\partial f}{\partial x^i} a^i \cdot \frac{dx}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x^i} a^i \cdot \frac{dx}{dt}$$

$$= \frac{\partial f}{\partial x^i} a^i \cdot \frac{dx^j}{dt} a_j$$

$$= \frac{\partial f}{\partial x^i} a^i \cdot x'(t)$$

$$= \nabla f \cdot x'(t)$$

$$= u \cdot x'(t)$$

$$\int_C u(x) \cdot dx = \int_{t_a}^{t_b} \frac{df(x(t))}{dt} = f(x_b) - f(x_a)$$

7.10

$\underline{u}(\underline{x}) = \nabla f(\underline{x})$

$$\int_C \underline{u}(\underline{x}) \cdot d\underline{x} = \int_C \nabla f(\underline{x}) \cdot d\underline{x}$$ [illegible]

[illegible]

$$= \int_C df$$

$$= \int_C [illegible] \frac{df}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x^i} a^i \cdot dx \cdot \frac{1}{dt}$$

7.9

a sphere of radius R has area $4\pi R^2$, hence $A = 4\pi R^2$, or

$$Q(R) = -\int_{A=4\pi R^2} \lambda \frac{\partial T}{\partial n} dA = -\int_{A=4\pi R^2} \lambda \nabla T \cdot n \, dA$$

$$\nabla T = \frac{\partial T}{\partial r} \underline{e}_r(\theta, \phi)$$

$$= -4\pi R^2 \lambda \left.\frac{\partial T}{\partial r}\right|_R$$

, as $\underline{n}$ is in the same direction as r, hence $\underline{n} = \underline{e}_r$, so $\nabla T \cdot \underline{n} = \frac{\partial T}{\partial r}$

③ parabolic temperature profile

$$Q = -4\pi R^2 \lambda \left.\frac{\partial\left(T_R + (T_0 - T_R)\left(1 - \frac{r^2}{R^2}\right)\right)}{\partial r}\right|_R =$$

$$-4\pi\lambda R^2 \left.\left(\frac{2T_R r}{R^2} - \frac{2T_0 r}{R^2}\right)\right|_R =$$

$$-8\pi\lambda\left.\left(T_R r - T_0 r\right)\right|_R =$$

$$-8\pi\lambda\left(T_R R - T_0 R\right) =$$

$$8\pi\lambda R\left(T_0 - T_R\right)$$

7.7 The unit of $\frac{\partial T}{\partial \underline{u}}$ is $\frac{\mathbf{K}}{m}$

7.8

$$\frac{\partial f}{\partial \underline{e}_r} = \nabla f \cdot \underline{e}_r = \frac{\partial f}{\partial r}$$

$$\frac{\partial f}{\partial \underline{e}_\theta} = \nabla f \cdot \underline{e}_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$$

$$\frac{\partial f}{\partial \underline{e}_\phi} = \nabla f \cdot \underline{e}_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$$

7.9

$$\underline{q} = -\lambda \nabla T$$

Through a given surface element dA, we know that the flux density is given by

$$\underline{q} \cdot d\underline{A} = -\lambda \nabla T \cdot d\underline{A} = -\lambda \nabla T \bullet \underline{n}\, dA = -\lambda \frac{\partial T}{\partial \underline{n}}\, dA$$

Integrating over the whole surface A gives

$$Q = -\int_A \lambda \frac{\partial T}{\partial \underline{n}}\, dA$$

$\boxed{7.4}$

$$dT = \frac{\partial T}{\partial t}\, dt + \nabla T \cdot d\underline{x}$$

dividing by dt gives

$$\frac{dT(t, \underline{x}(t))}{dt} = \frac{\partial T}{\partial t} + \nabla T \cdot \frac{d\underline{x}}{dt} = \frac{\partial T}{\partial t} + \nabla T \cdot \underline{V}$$

$\boxed{7.5}$

$$\nabla f(\underline{x}(x^1, x^2, x^3)) = \frac{\partial f}{\partial x^i}\, a^i = \frac{1}{h_i} \frac{\partial f}{\partial x^i}\, e^{(i)}$$

$e^{(i)} =$ [illegible] $h_i a^i = \frac{1}{h_i} a_i$ [illegible]

$a^i = \frac{1}{h_i} e^{(i)}$

Cartesian: $\nabla f(\underline{x}(x, y, z)) = \frac{\partial f}{\partial x} \underline{e}_x + \frac{\partial f}{\partial y} \underline{e}_y + \frac{\partial f}{\partial z} \underline{e}_z$ as $h_x = h_y = h_z = 1$

cylindrical: $\nabla f(\underline{x}(r, \phi, z)) = \frac{\partial f}{\partial r} \underline{e}_r + \frac{1}{r} \frac{\partial f}{\partial \phi} \underline{e}_\phi + \frac{\partial f}{\partial z} \underline{e}_z$

spherical: $\nabla f(\underline{x}(r, \theta, \phi)) = \frac{\partial f}{\partial r} \underline{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \underline{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \underline{e}_\phi$

$\boxed{7.6}$

~~$\frac{dT}{dt} = \nabla T \cdot x'(t)$~~

~~$= \nabla T \cdot v_r \underline{e}_r$~~

$\nabla T = \frac{dT}{dr} \underline{a}^r (\theta, \phi)$ (as T is a function only r)

$x'(t) = v_r \underline{a}_r$

$$\frac{dT}{dt} = \nabla T \cdot x'(t) = \frac{dT}{dr} \underline{a}^r \cdot v_r \underline{a}_r = \frac{dT}{dr} v_r$$

$x'(t) = \omega_\phi a_\phi$

$$\frac{dT}{dt} = \frac{dT}{dr} \underline{a}^r \cdot \omega_\phi \underline{a}_\phi = 0$$

7.2

$$dx = a_j \cdot dx^j$$

$$df = \frac{\partial f}{\partial x^i} a^i \cdot dx$$

$$df = \frac{\partial f}{\partial x^i} a^i \cdot a_j \, dx^j$$

$$= \frac{\partial f}{\partial x^i} \delta^i_j \, dx^j$$

$$= \frac{\partial f}{\partial x^i} dx^i$$

7.3

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x^i} dx^i$$

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x^i} \delta^i_j \, dx^j$$

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x^i} a^i \cdot a_j \, dx^j$$

$$= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x^i} a^i \cdot d\underset{\sim}{x}$$

$$= \frac{\partial f}{\partial t} dt + \nabla f \cdot d\underline{x}$$

$\boxed{7.1}$

~~$df = \frac{\partial f}{\partial x^i} dx^i$~~ $\longrightarrow dx^i = \delta^i_j dx^j$

~~$= \frac{\partial f}{\partial x^i} \delta^i_i dx^i$~~

~~$= \frac{\partial f}{\partial x^i} a^i \cdot a_i dx^i$~~

~~$= \frac{\partial f}{\partial x^i} a^i \cdot dx$~~

$$\frac{\partial f}{\partial x^i} a^i \cdot dx =$$

$$\frac{\partial f}{\partial x^i} a^i \cdot a_j dx^j =$$

$$\frac{\partial f}{\partial x^i} \delta^i_j dx^j =$$

$$\frac{\partial f}{\partial x^i} dx^i = df$$

$\boxed{7.2}$

$$f(\underline{x}(x^1, x^2, x^3))$$

$$\underline{a}_i = \frac{\partial}{\partial x^i} \underline{x}$$

$$\underline{a}^i \cdot \underline{a}_j = \delta^i_j$$

$$dx = \underline{a}_j dx^j \quad (= dx^j \underline{a}_j)$$

$$df = \frac{\partial f}{\partial x^i} \underline{a}^i \cdot d\underline{x}$$

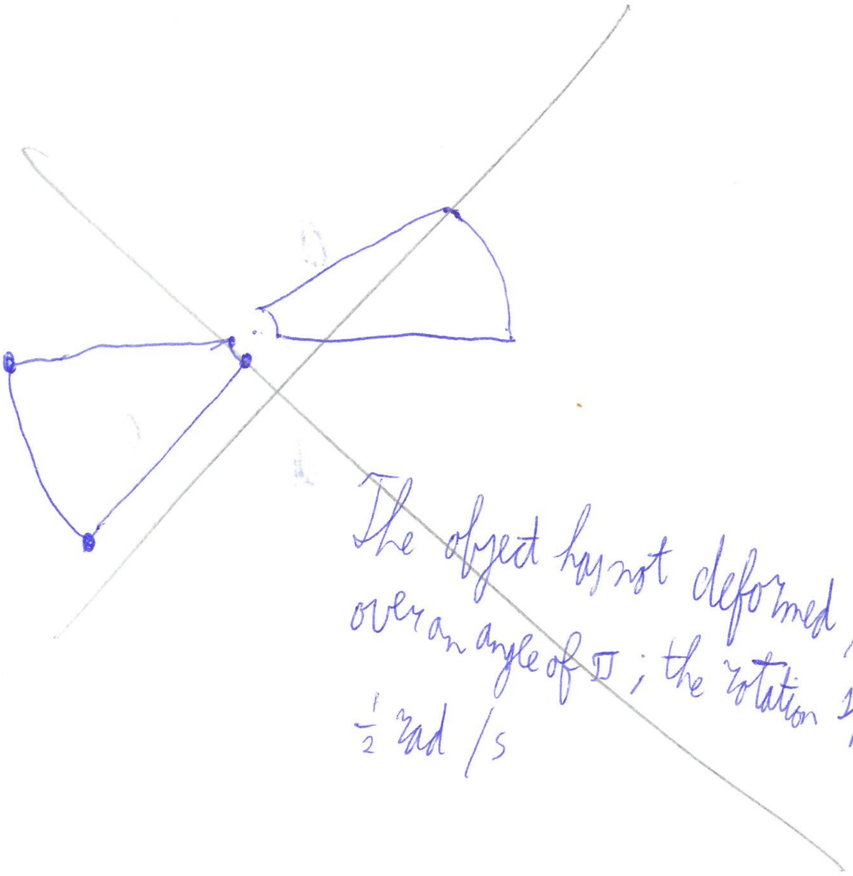
$$df = \frac{\partial f}{\partial x^i} a^i \cdot dx$$

$$= \frac{\partial f}{\partial x^i} a^i \cdot a_j dx^j$$

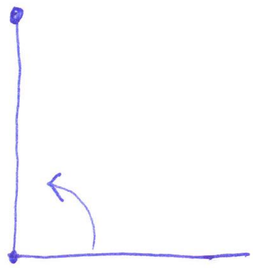
$$= \frac{\partial f}{\partial x^i} \delta^i_j dx^j$$

$$= \frac{\partial f}{\partial x^i} dx^i$$

8.13



The object has not deformed, but it has rotated over an angle of π; the rotation speed is $\frac{1}{2}$ rad/s



not deformed; but rotated 90° or $\frac{\pi}{2}$ rad; rotation speed = angular velocity = ω

8.12

$$\nabla \times \nabla f =$$

$$\nabla \times \left(\frac{\partial f}{\partial x^i} \underline{a}^i \right) =$$

$$\text{"}f_2\text{"} = \frac{\partial f}{\partial x^2} \underline{a}^2$$

$$\frac{1}{\sqrt{g}} \left(\frac{\partial^2 f}{\partial x^2 \partial x^3} - \frac{\partial^2 f}{\partial x^3 \partial x^2} \right) \underline{a}_1 +$$

$$\frac{1}{\sqrt{g}} \left(\frac{\partial^2 f}{\partial x^3 \partial x^1} - \frac{\partial^2 f}{\partial x^1 \partial x^3} \right) \underline{a}_2 +$$

$$\frac{1}{\sqrt{g}} \left(\frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{\partial^2 f}{\partial x^2 \partial x^1} \right) \underline{a}_3 = \underline{0}$$

8.10

$$\nabla \cdot (\nabla \times \underline{V}) = \nabla \cdot \left(\frac{1}{\sqrt{g}} \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) \underline{a}_1 + \frac{1}{\sqrt{g}} \left(\frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) \underline{a}_2 + \frac{1}{\sqrt{g}} \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \underline{a}_3 \right)$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) \underline{a}_1 + \left(\frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) \underline{a}_2 + \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \underline{a}_3 \right) =$$

$$\frac{1}{\sqrt{g}} \left(\frac{\partial^2 E_3}{\partial x^1 \partial x^2} - \frac{\partial^2 E_2}{\partial x^1 \partial x^3} + \frac{\partial^2 E_1}{\partial x^2 \partial x^3} - \frac{\partial^2 E_3}{\partial x^1 \partial x^2} + \frac{\partial^2 E_2}{\partial x^1 \partial x^3} - \frac{\partial^2 E_1}{\partial x^2 \partial x^3} \right) = \frac{1}{\sqrt{g}} \cdot 0$$

$$= 0$$

8.11 taking divergence of both sides of eq 8.15:

$$\nabla \times \underline{E}(t, x) = -\frac{\partial}{\partial t} \underline{B}(t, x)$$

$$\nabla \cdot (\nabla \times \underline{E}(t, x)) = -\frac{\partial}{\partial t} (\nabla \cdot \underline{B}(t, x))$$

$$0 = -\frac{\partial}{\partial t} (\nabla \cdot \underline{B}(t, x))$$

This implies that $\nabla \cdot \underline{B}$ is constant, e.g. if $\nabla \cdot \underline{B} = 0$ at one point in time (such as beginning of time), then it will always be zero

8.9 ①

$$\nabla \times \underline{V}(\underline{x}(x,y,z)) = \frac{\partial V_x}{\partial x}\underline{e}_{(y)} = \frac{V}{Z}\underline{e}_{(y)}$$

② $$\nabla \times \underline{V}(\underline{x}(r,\phi,z)) = -\frac{\partial V_z}{\partial r}\underline{e}_{(\phi)}$$ ~~$+\frac{1}{r}\left(\frac{\partial r V_\phi}{\partial r}\right)e_{(z)}$~~ ~~$\frac{2V_r}{R^2}e_{(\phi)}$~~

$$= \frac{2Vr}{R^2}\underline{e}_{(\phi)}$$

③ $$\nabla \times \underline{V}(\underline{x}(r,\phi,z)) = \frac{1}{r}\frac{\partial r V_\phi}{\partial r}\underline{e}_{(z)} = 2w\,\underline{e}_{(z)}$$

④ $$\nabla \times \underline{V}(\underline{x}(r,\phi,z)) = \frac{1}{r}\frac{\partial r V_\phi}{\partial r}\underline{e}_{(z)} = \frac{V}{r}\underline{e}_{(z)}$$

⑤ $$\nabla \times \underline{V}(\underline{x}(r,\phi,z)) = \frac{1}{r}\frac{\partial r V_\phi}{\partial r}e_{(z)} = \underline{0}$$

8.8 ④

$$\nabla \times \underline{v}\left(\underline{x}(r,\theta,\phi)\right) = \frac{1}{r^2 \sin\theta}\left(\frac{\partial\, r\sin\theta\, v_\phi}{\partial\theta} - \frac{\partial\, r\, v_\theta}{\partial\phi}\right)\underline{e}_{(r)} +$$

$$\frac{1}{r\sin\theta}\left(\frac{\partial\, v_r}{\partial\phi} - \frac{\partial\, r \sin\theta\, v_\phi}{\partial r}\right)\underline{e}_{(\theta)} +$$

$$\frac{1}{r}\left(\frac{\partial\, r\, v_\theta}{\partial r} - \frac{\partial\, v_r}{\partial\theta}\right)\underline{e}_{(\phi)} =$$

$$\frac{1}{r\sin\theta}\left(\frac{\partial \sin\theta\, v_\phi}{\partial\theta} - \frac{\partial\, v_\theta}{\partial\phi}\right)\underline{e}_{(r)} +$$

$$\frac{1}{r}\left(\frac{1}{\sin\theta}\,\frac{\partial v_r}{\partial\phi} - \frac{\partial\, r\, v_\phi}{\partial r}\right)\underline{e}_{(\theta)} +$$

$$\frac{1}{r}\left(\frac{\partial\, r\, v_\theta}{\partial r} - \frac{\partial\, v_r}{\partial\theta}\right)\underline{e}_{(\phi)}$$

$\boxed{8.8}$ ③

$$\nabla \times \underline{V}(\underline{x}(r,\phi,z)) = \frac{1}{r}\left(\frac{\partial 1 \cdot V_z}{\partial \phi} - \frac{\partial r\, V_\phi}{\partial z}\right)e_{(r)} +$$

$$\frac{1}{1}\left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)e_{(\phi)} +$$

$$\frac{1}{r}\left(\frac{\partial r V_\phi}{\partial r} - \frac{\partial V_r}{\partial \phi}\right)e_{(z)} =$$

$$\left(\frac{1}{r}\frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z}\right)e_{(r)} + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)e_{(\phi)} +$$

$$\frac{1}{r}\left(\frac{\partial r V_\phi}{\partial r} - \frac{\partial V_r}{\partial \phi}\right)e_{(z)}$$

∂.∂) ②

$$\nabla \times \underline{V}(\underline{x}(x,y,z)) = \frac{1}{1\cdot 1}\left(\frac{\partial 1\cdot V_z}{\partial y} - \frac{\partial 1\cdot V_y}{\partial z}\right)\underline{e}_{(x)} +$$

$$\frac{1}{1\cdot 1}\left(\frac{\partial 1\cdot V_x}{\partial z} - \frac{\partial 1\cdot V_z}{\partial x}\right)\underline{e}_{(y)} +$$

$$\frac{1}{1\cdot 1}\left(\frac{\partial 1\cdot V_y}{\partial x} - \frac{\partial 1\cdot V_x}{\partial y}\right)e_{(z)} =$$

$$\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)e_{(x)} +$$

$$\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)e_{(y)} +$$

$$\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)e_{(z)}$$

q.e.d.

8.8 | ①

$$\nabla \times \underline{E} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial h_3 E_{(3)}}{\partial x^2} - \frac{\partial h_2 E_{(2)}}{\partial x^3}\right) h_1 \underline{e}_{(1)} +$$

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial h_1 E_{(1)}}{\partial x^3} - \frac{\partial h_3 E_{(3)}}{\partial x^1}\right) h_2 \underline{e}_{(2)} +$$

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial h_2 E_{(2)}}{\partial x^1} - \frac{\partial h_1 E_{(1)}}{\partial x^2}\right) h_3 \underline{e}_{(3)} =$$

$$\frac{1}{h_2 h_3}\left(\frac{\partial h_3 E_{(3)}}{\partial x^2} - \frac{\partial h_2 E_{(2)}}{\partial x^3}\right) \underline{e}_{(1)} +$$

$$\frac{1}{h_1 h_3}\left(\frac{\partial h_1 E_{(1)}}{\partial x^3} - \frac{\partial h_3 E_{(3)}}{\partial x^1}\right) \underline{e}_{(2)} +$$

$$\frac{1}{h_1 h_2}\left(\frac{\partial h_2 E_{(2)}}{\partial x^1} - \frac{\partial h_1 E_{(1)}}{\partial x^2}\right) \underline{e}_{(3)}$$

8.5

$$\nabla \times \underline{E} = \frac{1}{\sqrt{g}} e^{ijk} \frac{\partial E_k}{\partial x^j} \underline{a}_i =$$

$$\frac{1}{\sqrt{g}} \left(\frac{\partial E_3}{\partial x^2} \underline{a}_1 - \frac{\partial E_2}{\partial x^3} \underline{a}_1 + \frac{\partial E_1}{\partial x^3} \underline{a}_2 - \frac{\partial E_3}{\partial x^1} \underline{a}_2 + \frac{\partial E_2}{\partial x^1} \underline{a}_3 - \frac{\partial E_1}{\partial x^2} \underline{a}_3 \right)$$

which will lead to eq. (8.12) q.e.d

$$= \frac{1}{\sqrt{g}} \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) \underline{a}_1 + \frac{1}{\sqrt{g}} \left(\frac{\partial E_1}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) \underline{a}_2 + \frac{1}{\sqrt{g}} \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \underline{a}_3$$

8.6

$$\nabla \times \underline{E}(t, x) = \frac{1}{\sqrt{g}} \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) \underline{a}_1 + \frac{1}{\sqrt{g}} \left(\frac{\partial E_{\text{[illegible]}}}{\partial x^3} - \frac{\partial E_3}{\partial x^1} \right) \underline{a}_2 +$$

$$\frac{1}{\sqrt{g}} \left(\frac{\partial E_3}{\partial x^2} - \frac{\partial E_2}{\partial x^3} \right) \underline{a}_3 = s^1 \underline{a}_1 + s^2 \underline{a}_2 + s^3 \underline{a}_3 = s^i \underline{a}_i$$

$$= \frac{-\partial}{\partial t} \beta(t, x) \qquad \square$$

8.7

$$\oint_{\partial A} \underline{E}(t, \underline{x}) \cdot d\underline{x} = \int_A \nabla \times \underline{E}(t, \underline{x}) \cdot d\underline{A}$$

$$\oint_{\partial A} \underline{v} \cdot d\underline{x} = \int_A (\nabla \times \underline{v}) \, d\underline{A}$$

8.4

$$\oint_{\partial A} E \cdot dx = \int_A s \cdot dA$$

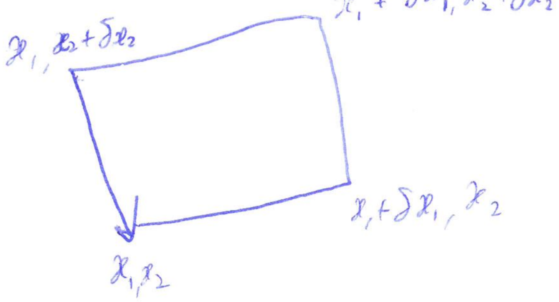
not $\frac{1}{\sqrt{g}}$! $\rightarrow$ $\sqrt{g}$

$$dA = \delta A_3 = a^3 \delta x^1 \delta x^2$$

$$= \int_A s^i a_i \cdot a^3 \delta x^1 \delta x^2 \sqrt{g}$$

$$= \int_A s^3 \delta x^1 \delta x^2 \sqrt{g}$$

$$= \int_{\partial A} E_i dx^i$$



$$= E(x^1, x^2, x^3)\, a^i \cdot a_1 \,\delta x^1 + E(x^1, x^2 + \delta x^2, x^3)\, a^i \cdot -a_1 \,\delta x^1$$

$$+ E(x^1 + \delta x^1, x^2, x^3)\, a^i \cdot a_2 \,\delta x^2 + E(x^1, x^2, x^3)\, a^i \cdot a_2 \,\delta x^2 \cdot -1$$

$$= \frac{\partial E_1}{\partial x^2} \delta x^1 \delta x^2 \cdot -1 + \frac{\partial E_2}{\partial x^1} \delta x^1 \delta x^2$$

$$= \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right) \delta x^1 \delta x^2$$

$$s^3 = \frac{1}{\sqrt{g}} \left(\frac{\partial E_2}{\partial x^1} - \frac{\partial E_1}{\partial x^2} \right)$$

$\boxed{8.3}$

$$\phi_1 = \int_{A_1} \underline{B}\cdot d\underline{A} = \int_{A_1} \left(\underline{B}\cdot\underline{n}_c\right) dA = \int_{A_1} \left(\underline{B}\cdot\underline{n}_a\right) dA$$

$$\phi_2 = \int_{A_2} \underline{B}\cdot d\underline{A} = \int_{A_2} \left(\underline{B}\cdot\underline{n}_c\right) dA = \int_{A_2} -\left(\underline{B}\cdot\underline{n}_a\right) dA = -\int_{A_2} \left(\underline{B}\cdot\underline{n}_a\right) dA$$

$$0 = \phi = \int_{A_1+A_2} \left(\underline{B}\cdot d\underline{A}\right) = \int_{A_1+A_2} \left(\underline{B}\cdot\underline{n}\right) dA = \int_{A_1} \left(\underline{B}\cdot\underline{n}\right) dA + \int_{A_2} \left(\underline{B}\cdot\underline{n}\right) dA$$

$$= \phi_1 - \phi_2$$

8.1

If it were the case that E can be written as the gradient of a scalar field, the the integral on the left hand side of eq (8.4) will evaluate to zero. But, the right-hand side of the equation, which is the change of magnetic flux p.u.t (per unit of time), generally is not equal to zero. This leads to a contradiction.

8.2

$$\nabla \cdot B = 0 = \lim_{V \to 0} \frac{1}{V} \int_{\partial V} (\underline{\Gamma}_B \cdot \underline{n}) dA$$

Divergence theorem:

$$\int_{\partial V} (\underline{\Gamma}_s \cdot \underline{n}) dA = \int_V \nabla \cdot \underline{\Gamma} \, dV$$

$\nabla \cdot B = 0$ and $(\underline{B} \cdot \underline{n}) dA = \underline{B} d \underline{A}$, hence

$$\int_{\partial V} \underline{B} d \underline{A} = \int_{\partial V} (\underline{B} \cdot \underline{n}) dA = \int_V \nabla \cdot B \, dV = \int 0 \, dV = 0$$

∂V is a closed surface surrounding volume V; hence the magnetic flux $\int_{\partial V} \underline{B} d \underline{A}$ through a closed surface ∂V is zero.

$\square$

9.1

$$\nabla \cdot (f \nabla \times \underline{V}) = f(\nabla \cdot (\nabla \times \underline{V})) + (\nabla f) \cdot (\nabla \times \underline{V})$$

$$= f\,(0) + (\nabla f) \cdot (\nabla \times \underline{V})$$

$$= (\nabla f) \cdot (\nabla \times \underline{V}) \qquad \square$$

9.2

$$\left.\begin{aligned} \nabla f &= \frac{\partial f}{\partial x^i}\, \underline{a}^i \\ g^{ji}\, \underline{a}_i &= \underline{a}^j \end{aligned}\right\} \nabla f = \frac{\partial f}{\partial x^j}\, g^{ji}\, \underline{a}_i$$

how to explain this change $(i \to j)$

where did it go?

$$= \Gamma^i$$

$$\nabla^2 f = \nabla \cdot \nabla f$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ji} \frac{\partial f}{\partial x^j}\right)$$

9.3

$$= (g_{ji})^{-1}$$

$$\nabla^2 f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ji} \frac{\partial f}{\partial x^j}\right)$$

$$= \frac{1}{h_1 h_2 h_3} \underbrace{\frac{\partial}{\partial x^i}}_{\sum_{i=1}^{3}} \left(h_1 h_2 h_3\, g^{ii} \frac{\partial f}{\partial x^i}\right) = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x^1}\left(h_1 h_2 h_3 \cdot h_1^{-2} \frac{\partial f}{\partial x^1}\right)\right) +$$

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x^2}\left(h_1 h_2 h_3 \cdot h_2^{-2} \frac{\partial f}{\partial x^2}\right)\right) +$$

$$\frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x^3}\left(h_1 h_2 h_3 \cdot h_3^{-2} \frac{\partial f}{\partial x^3}\right)\right)$$

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x^1}\left(\frac{h_2 h_3}{h_1} \frac{\partial f}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_1 h_3}{h_2} \frac{\partial f}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1 h_2}{h_3} \frac{\partial f}{\partial x^3}\right)\right)$$

$\boxed{9.4}$ Cartesian: $h_x = h_y = h_z = 1$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Cylindrical: $h_r = 1 = h_z$

$h_\phi = r$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Spherical: $h_r = 1$ $h_\theta = r$

$h_\phi = r\sin\theta$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 f}{\partial \phi^2}$$

$\boxed{9.5}$

$\|x\|^2 = x^2 + y^2 + z^2$ in Cartesian

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial x^2} = 2 + 2 + 2 = 6$$

$x = r\underline{e}_r(r,\phi) + z e_z \longrightarrow \|x\|^2 = r^2 + z^2$

$$\nabla^2 f = \cancel{\frac{\partial^2 f}{\partial r^2} + \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} +} \; \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(2r^2) + 2$$

$$= \frac{1}{r}4r + 2 = 4 + 2$$

$$= 6$$

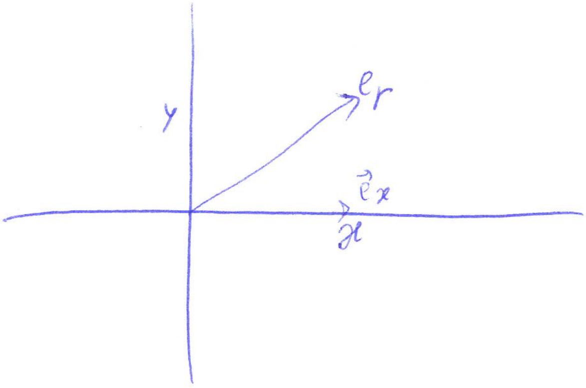
$\vec{e}_z = \vec{e}_z$

$e_x = r\cos\phi e$

$\vec{e}_r = x\vec{e}_x + \vec{y}e_y$

$e_\phi =$

$$\nabla^2 a_x = \frac{\partial^2 a_x}{\partial x^2} + \frac{\partial^2 a_x}{\partial y^2} + \frac{\partial^2 a_x}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\right)$$



$x = r\cos\phi$
$y = r\sin\phi$
$z = z$

$$e_r = \frac{\partial}{\partial r} r\cos\phi\, e_x + \frac{\partial}{\partial r} r\sin\phi\, e_y$$

$$e_\phi = C\left(\frac{\partial}{\partial\phi} r\cos\phi\, e_x + \frac{\partial}{\partial\phi} r\sin\phi\, e_y\right)$$

$$= C(-\sin\phi\, r\, e_x + \cos\phi\, r e_y) = -\sin\phi\, e_x + \cos\phi\, e_y$$

$$C = \frac{1}{\sqrt{r^2\sin^2\phi + r^2\cos^2\phi}} = \frac{1}{r}$$

$$\begin{pmatrix} e_r \\ e_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} e_x \\ e_y \end{pmatrix}$$